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Asymptotic Behavior of Polynomials Satisfying Three-Term Recurrence Relations

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Sufficient conditions are given to ensure that

$$\lim_{n\to\infty} B_n\left(\frac{z}{n+1}\right)=e^{\gamma z} \qquad \text{for all} \quad z\in\mathbb{C},$$

where the $B_n(z)$ are defined by three-term recurrence relations

$$B_{-1}(z):=0,\ B_0(z):=1,\ B_n(z):=(1+\beta_n z)\,B_{n-1}(z)+\alpha_n z^\lambda B_{n-2}(z), \qquad n\geqslant 1$$

and hence are nth denominators of continued fractions

$$\frac{\alpha_1 z^\lambda}{1+\beta_1 z}+\frac{\alpha_2 z^\lambda}{1+\beta_2 z}+\frac{\alpha_3 z^\lambda}{1+\beta_3 z}+\cdots, \qquad \lambda=1 \text{ or } 2,\ 0\neq\alpha_n\in\mathbb{C},\ \beta_n\in\mathbb{C}.$$

Here $\gamma:=(2-\lambda)\,\alpha+\beta$, where $\alpha:=\lim\alpha_n$ and $\beta:=\lim\beta_n$. In addition to proving uniform convergence on compact subsets of $\mathbb{C}$, we obtain explicit information on the order of convergence of the sequences $\{B_n(z/(n+1))\}$ and $\{p_k^{(n)}\}_{n=1}^\infty$, where $\sum_{k=0}^{d_n} p_k^{(n)} z^k := B_n(z/(n+1))$. Important types of continued fractions subsumed under the above class include regular C-fractions, general T-fractions, and associated continued fractions, all three of which have their approximants in Padé tables. Since J-fractions are essentially equivalent to associated continued fractions, many of our results describe the asymptotic behavior of orthogonal polynomial sequences.

1. INTRODUCTION

Polynomial sequences $\{P_n(z)\}$ satisfying three-term recurrence relations of the form

$$P_n(z)=b_n(z)\,P_{n-1}(z)+a_n(z)\,P_{n-2}(z), \qquad n=1,2,3,..., \tag{1.1}$$

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where the $a_n(z)$ and $b_n(z)$ are polynomials in z of degree at most two arise in the study of orthogonal polynomials [3, 5, 12, 14] and continued fractions [9]. With suitable initial conditions

$$A_{-1}(z)=1, A_0(z)=0 \qquad \text{or} \qquad B_{-1}(z)=0, B_0(z)=1, \tag{1.2}$$

$P_n(z)$ becomes the nth numerator $A_n(z)$ or denominator $B_n(z)$ of the continued fraction

$$\mathop{K}_{n=1}^{\infty}\left(\frac{a_n(z)}{b_n(z)}\right)=\frac{a_1(z)}{b_1(z)+}\,\frac{a_2(z)}{b_2(z)+}\,\frac{a_3(z)}{b_3(z)+}\cdots. \tag{1.3}$$

One of the authors [15] has recently proved that if

$$\sum_{k=1}^{\infty}|a_k(z)-a(z)|<\infty \qquad \text{and} \qquad \sum_{k=1}^{\infty}|b_k(z)-b(z)|<\infty \tag{1.4}$$

then the sequences

$$\left\{\frac{A_n(z)}{(-x_2(z))^{n+1}}\right\} \qquad \text{and} \qquad \left\{\frac{B_n(z)}{(-x_2(z))^{n+1}}\right\} \tag{1.5}$$

converge to holomorphic functions. Here $x_2(z)$ is determined by the condition that $|x_1(z)/x_2(z)|<1$, where $x_1(z)$ and $x_2(z)$ are fixed points of the transformation $T(w)=a(z)/(b(z)+w)$; that is, solutions of $w^2+b(z)\,w-a(z)=0$. This phenomenon is called *separate convergence* since both sequences (1.5) converge and hence the approximant sequence $\{A_n(z)/B_n(z)\}$ of (1.3) is convergent. Recent work on separate convergence (see, e.g., [7, 11, 15–17] and references contained therein) is closely related to the study of asymptotic properties of orthogonal polynomials [14, Chap. VIII; 19, 20].

In the present paper we are concerned with the behavior of the sequences

$$\left\{A_n\left(\frac{z}{n+1}\right)\right\} \qquad \text{and} \qquad \left\{B_n\left(\frac{z}{n+1}\right)\right\}, \tag{1.6}$$

where $A_n(z)$ and $B_n(z)$ denote the nth numerator and denominator, respectively, of continued fractions of the form

$$\mathop{K}_{n=1}^{\infty}\left(\frac{\alpha_n z^{\lambda}}{1+\beta_n z}\right), \qquad 0\neq\alpha_n\in\mathbb{C},\ \beta_n\in\mathbb{C},\ \lambda=1 \text{ or } 2. \tag{1.7}$$

If we wanted to prove only the convergence of the sequences (1.6), this could be done in some cases more easily than in the approach employed in this paper. For example, in the case in which (1.4) holds, one can use the convergence result in [15] applied to (1.5) and the convergence of $\{[-x_2(z/(n+1))]^{n+1}\}$ to deduce locally uniform convergence of (1.6). However, we know of no way to extend this to the other three cases dealt with (in Theorems 4.1 and 4.2) where (1.4) does not hold. More importantly, our goal is not only to prove convergence of (1.6) but also to obtain explicit estimates of the speed of convergence of the functions $B_n(z/(n+1))$ and of their coefficients.

Consideration is given to continued fractions (1.7) which are limit-periodic,

$$\lim_{k\to\infty} \alpha_k = \alpha \in \mathbb{C} \qquad \text{and} \qquad \lim_{k\to\infty} \beta_k = \beta \in \mathbb{C}, \tag{1.8}$$

and to those that are partly limit-periodic, satisfying

$$|\alpha_k| \leqslant Ek^\tau,\ E>0,\ k\geqslant 1 \qquad \text{and} \qquad \lim_{k\to\infty} \beta_k = \beta \in \mathbb{C}, \tag{1.9}$$

so that unbounded sequences $\{\alpha_k\}$ are permitted. For both cases (1.8) and (1.9) a unique pair (α, β) of complex numbers is determined by setting $\alpha=0$ if (1.9) holds but not (1.8). $x_1(z)$ and $x_2(z)$ denote the roots of the quadratic equation

$$w^2 + (1+\beta z)\,w - \alpha z^\lambda = 0 \tag{1.10}$$

given by

$$-x_m(z) = \frac{1+\beta z}{2}\left[1 + (-1)^m \sqrt{1 + \frac{4\alpha z^\lambda}{(1+\beta z)^2}}\,\right], \qquad m=1,2. \tag{1.11}$$

Since no branch point of the expression $\sqrt{\ }$ in (1.11) occurs at $z=0$, a region D containing a neighborhood of the origin exists and a branch of $\sqrt{\ }$ exists such that Re $\sqrt{\ } > 0$ for all $z\in D$. Hence the functions $x_m(z)$ are well defined for $z \in D$ and satisfy

$$\left|\frac{x_1(z)}{x_2(z)}\right| < 1 \qquad \text{for} \quad z\in D. \tag{1.12}$$

In the sequel it is convenient to let $n_0(R)$ denote the least integer satisfying

$$\frac{z}{n} \in D \qquad \text{for all} \quad |z| \leqslant R \text{ and } n \geqslant n_0(R). \tag{1.13}$$

We also employ the notation

$$\gamma := (2-\lambda)\,\alpha + \beta, \qquad \Gamma := \tfrac{1}{2}[(-1)^\lambda\, 2\alpha(\alpha+\beta)^{2-\lambda} - \gamma^2],$$

$$\Delta := \sum_{k=2}^{\infty} (\alpha_k - \alpha), \qquad \Theta := \sum_{k=1}^{\infty} (\beta_k - \beta), \tag{1.14}$$

when the series in question converge. One of the main results of the paper (Theorem 4.1 (A)) is that, if

$$\sum_{k=1}^{\infty} |\alpha_k - \alpha| < \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} |\beta_k - \beta| < \infty, \tag{1.15}$$

then

$$B_n\left(\frac{z}{n+1}\right) - e^{\gamma z} = e^{\gamma z}\left[\frac{\Gamma z^2 + (\Theta - \beta) z}{n} + \frac{(\Delta - 2\alpha) z^{\lambda}}{n^{\lambda}}\right] + o\left(\frac{1}{n}\right), \tag{1.16}$$

where $o(1/n) \to 0$ as $n \to \infty$, uniformly for $|z| \leqslant R$. From this it is shown (Theorem 4.2 (A)) that, if $p_k^{(n)}$ are coefficients defined by

$$B_n\left(\frac{z}{n+1}\right) =: \sum_{k=0}^{d_n} p_k^{(n)} z^k, \qquad d_n \leqslant n,\ n = 1, 2, 3, ..., \tag{1.17}$$

then

$$p_0^{(n)} = 1, \qquad p_1^{(n)} = \gamma + \frac{(2-\lambda)(\Delta - 2\alpha) + (\Theta - \beta)}{n} + o\left(\frac{1}{n}\right), \tag{1.18a}$$

$$p_k^{(n)} = \frac{\gamma^k}{k!} + \frac{\gamma^{k-2}[(k-1)\,\Gamma + \gamma(\Theta - \beta)]}{n(k-1)!}$$

$$+ \frac{\gamma^{k-\lambda}(\Delta - 2\alpha)}{n^{\lambda}(k-\lambda)!} + o\left(\frac{1}{n}\right), \qquad 2 \leqslant k \leqslant d_n. \tag{1.18b}$$

We are therefore able to give not only the limiting values of $\{B_n(z/(n+1))\}_{n=1}^{\infty}$ and $\{p_k^{(n)}\}_{n=1}^{\infty}$, but also explicit information about their order of convergence. Since

$$b_k^{(n)} = (n+1)^k p_k^{(n)}, \qquad \text{where } \sum_{k=0}^{d_n} b_k^{(n)} z^k := B_n(z),$$

(1.18) yields knowledge about the coefficients $b_k^{(n)}$ of $B_n(z)$. Results similar to (1.16) and (1.18) are also given (Theorems 4.1 and 4.2(B)) under weaker conditions than (1.15). However, as expected, the information on the order of convergence is less explicit. By means of Theorem 4.4 one sees that all results for denominators $B_n(z)$ in Theorems 4.1 and 4.2 are applicable to the numerators $A_n(z)$ as well.

Our proof of Theorem 4.1 is based on properties of the functions $x_m(z)$ and $B_n(z)$ developed in Sections 2, 3, and 4. A central role is played by the recurrence relations

$$B_{-1}(z) := 0,\ B_0(z) := 1,\ B_n(z) := (1 + \beta_n z)\, B_{n-1}(z) + \alpha_n z^{\lambda} B_{n-2}(z), \quad n \geqslant 1, \tag{1.19}$$

that define the polynomials $B_n(z)$. Bounds for and convergence of the sequences $\{[x_m(z/n)]^{n-k}\}_{n=1}^{\infty}$ are derived in Section 2 where it is shown, in particular (Lemmas 2.1 and 2.2), that, for a fixed non-negative integer k,

$$\lim_{n\to\infty} x_1\left(\frac{z}{n}\right) = \lim_{n\to\infty}\left[-x_1\left(\frac{z}{n}\right)\right]^{n-k} = 0, \tag{1.20a}$$

$$\lim_{n\to\infty}\left[-x_2\left(\frac{z}{n}\right)\right] = 1, \qquad \lim_{n\to\infty}\left[-x_2\left(\frac{z}{n}\right)\right]^{n-k} = e^{\gamma z}. \tag{1.20b}$$

The main result of Section 3 gives a uniform bound for the sequences $\{B_k(z/(n+1))\}_{n=1}^{\infty}$, $1 \leqslant k < n$, $|z| \leqslant R$ (Lemma 3.1).

Three important types of continued fractions are subsumed under the form (1.7), all closely related to Padé tables. These continued fractions are of the forms

$$\mathop{K}_{n=1}^{\infty}\left(\frac{a_n z}{1}\right), \qquad \mathop{K}_{n=1}^{\infty}\left(\frac{F_n z}{1+G_n z}\right), \qquad \text{and} \qquad \mathop{K}_{n=1}^{\infty}\left(\frac{-K_n z^2}{1+L_n z}\right), \tag{1.21}$$

and are referred to in the literature as *regular C-fractions*, *general T-fractions*, and *associated continued fractions*, respectively. In Section 5 we describe the application of Theorems 4.1 and 4.2 to each of these three types of continued fractions. Examples are taken from families of hypergeometric functions ${}_2F_1(A, B; C; z)$ and ${}_1F_1(B; C; z)$. We also consider *real J-fractions*

$$\mathop{K}_{n=1}^{\infty}\left(\frac{-K_n}{L_n+\zeta}\right), K_1 = -1, K_n > 0 \quad \text{for} \quad n \geqslant 2$$
$$\text{and} \quad L_n \in R \quad \text{for} \quad n \geqslant 1, \tag{1.22}$$

whose denominators $\tilde{B}_n(\zeta)$ provide all orthogonal (monic) polynomial sequences. By establishing a simple relation between $\tilde{B}_n(\zeta)$ and the nth denominator $B_n(z)$ of the related associated continued fraction in (1.21), all convergence results derived for $\{B_n(z)\}$ can be applied to the sequence of orthogonal polynomials $\{\tilde{B}_n(\zeta)\}$. In this way we obtain asymptotic properties of orthogonal polynomials analogous to those found in [14, Chap. 8]. An example involving Legendre polynomials is included.

2. Convergence and Bounds for Sequences $\{[-x_m(z/n)]^{n-k}\}_{n=1}^{\infty}$

In this section we state and prove results about the roots $x_m(z)$ of the quadratic equation (1.10) that are subsequently used.

LEMMA 2.1. *Let $R>0$ and $k\geqslant 0$ be given. For $|z|\leqslant R$ and $n\geqslant n_0(R)$, let $f_1(z,n)$, $f_2(z,n,k)$, and $f_3(z,n,k)$ be defined by*

$$-x_2\left(\frac{z}{n}\right)=1+\frac{\gamma z}{n}+\frac{(-1)^\lambda\alpha(\alpha+\beta)^{2-\lambda}z^2}{n^2}+\frac{f_1(z,n)}{n^3}, \tag{2.1}$$

$$\left[-x_2\left(\frac{z}{n}\right)\right]^{n-k}=e^{\gamma z+(\Gamma z^2-k\gamma z)/n+f_2(z,n,k)/n^2}, \tag{2.2}$$

$$\left[-x_2\left(\frac{z}{n}\right)\right]^{n-k}-e^{\gamma z}=e^{\gamma z}\left[\frac{\Gamma z^2-k\gamma z}{n}+\frac{f_3(z,n,k)}{n^2}\right]. \tag{2.3}$$

Then there exist $n_2(R)$, $F_1(R)$, $F_2(R,k)$, $F_3(R,k)$ such that for $|z|\leqslant R$ and $n\geqslant n_2(R)$,

$$|f_1(z,n)|\leqslant F_1(R)<\infty,\ |f_\nu(z,n,k)|\leqslant F_\nu(R,k)<\infty, \qquad \nu=2,3. \tag{2.4}$$

Moreover, there exists a $K_1(R,k)$ such that

$$\left|\left[-x_2\left(\frac{z}{n}\right)\right]^{n-k}-e^{\gamma z}\right|\leqslant\frac{K_1(R,k)}{n} \qquad \text{for } |z|\leqslant R \quad \text{and} \quad n\geqslant n_2(R). \tag{2.5}$$

Proof. From the theory of Taylor series one has, for $|u|<1$ and $|v|<1$,

$$(1+u)^{1/2}=\sum_{j=0}^{\infty}\binom{1/2}{j}u^j=1+\tfrac{1}{2}u-\tfrac{1}{8}u^2+O(u^3), \qquad u\to 0, \tag{2.6}$$

and

$$\log(1+v)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}v^j}{j}=v-\tfrac{1}{2}v^2+O(v^3), \qquad v\to 0, \tag{2.7}$$

where the functions on the left are principal branches. Setting

$$u:=\frac{2\beta z}{n}+\frac{\beta^2z^2}{n^2}+\frac{4\alpha z^\lambda}{n^\lambda} \tag{2.8}$$

it is clear that there exists an $n_1(R)\geqslant n_0(R)$ such that $|u|\leqslant\frac{1}{2}$ for $|z|\leqslant R$ and $n\geqslant n_1(R)$. If $f(z,n)$ is defined by

$$-x_2\left(\frac{z}{n}\right)=1+\frac{\beta z}{n}+\frac{\alpha z^\lambda}{n^\lambda}-\frac{\alpha^2z^{2\lambda}}{n^{2\lambda}}-\frac{\alpha\beta z^{\lambda+1}}{n^{\lambda+1}}+\frac{f(z,n)}{n^3}, \tag{2.9}$$

then, using (1.11) and (2.6), we conclude that there exists an $F(R)$ satisfying

$$|f(z,n)|\leqslant F(R)<\infty \qquad \text{for } |z|\leqslant 1 \quad \text{and} \quad n\geqslant n_1(R).$$

If $\lambda = 1$, then (2.9) becomes

$$-x_2\left(\frac{z}{n}\right) = 1 + \frac{(\alpha+\beta)\,z}{n} - \frac{\alpha(\alpha+\beta)\,z^2}{n^2} + \frac{f(z,n)}{n^3}$$

and if $\lambda = 2$, it becomes

$$-x_2\left(\frac{z}{n}\right) = 1 + \frac{\beta z}{n} + \frac{\alpha z^2}{n^2} + \frac{f(z,n) - \alpha\beta z^3 - \alpha^2 z^4/n}{n^3}.$$

Thus with $f_1(z,n) := f(z,n)$ when $\lambda = 1$ and $f_1(z,n) := f(z,n) - \alpha\beta z^3 - \alpha^2 z^4/n$ when $\lambda = 2$, it follows that there exists a function $F_1(R)$ satisfying the first inequality in (2.4) for $|z| \leqslant R$ and $n \geqslant n_1(R)$.

Next we set

$$v := -x_2\left(\frac{z}{n}\right) - 1 = \frac{\gamma z}{n} + \frac{(-1)^\lambda\,\alpha(\alpha+\beta)^{2-\lambda}\,z^2}{n^2} + \frac{f_1(z,n)}{n^3} \tag{2.10}$$

and choose $n_2(R) \geqslant n_1(R)$ so that $|v| \leqslant \frac{1}{2}$ for $|z| \leqslant R$ and $n \geqslant n_2(R)$. Then if $g_\nu(z, n, k)$ are defined by

$$(n-k)\,v = \gamma z + \frac{(-1)^\lambda\,\alpha(\alpha+\beta)^{2-\lambda}\,z^2 - k\gamma z}{n} + \frac{g_1(z,n,k)}{n^2},$$

$$(n-k)\,v^2 = \frac{\gamma^2 z^2}{n} + \frac{g_2(z,n,k)}{n^2}, \qquad (n-k)\,O(v^3) = \frac{g_3(z,n,k)}{n^2},$$

it follows from (2.1) and (2.4) that there exist $G_\nu(R,k)$ such that

$$|g_\nu(z,n,k)| \leqslant G_\nu(R,k) \qquad \text{for} \quad |z| \leqslant R \qquad \text{and} \qquad n \geqslant n_2(R),\ \nu = 1, 2, 3.$$

Combining these results with (2.7) yields

$$\left[-x_2\left(\frac{z}{n}\right)\right]^{n-k} = (1+v)^{n-k} = e^{(n-k)\,v - (1/2)(n-k)\,v^2 + (n-k)\,O(v^3)}$$

from which we arrive at (2.5) and the second inequality in (2.4). ∎

LEMMA 2.2. *Let $R > 0$ and $k \geqslant 0$ be given. For $|z| \leqslant R$ and $n \geqslant n_0(R)$ let $f_4(z,n)$, $f_5(z,n)$, and $f_6(z,n,k)$ be defined by*

$$-x_1\left(\frac{z}{n}\right) = \frac{-(2-\lambda)\,\alpha z}{n} + \frac{f_4(z,n)}{n^2}, \tag{2.11}$$

$$\frac{1}{x_1(z/(n+1)) - x_2(z/(n+1))} = 1 - \frac{\beta z}{n} - \frac{2\alpha z^\lambda}{n^\lambda} + \frac{f_5(z,n)}{n^2}, \tag{2.12}$$

$$\left[-x_1\left(\frac{z}{n}\right)\right]^{n-k} = \left[\frac{-(2-\lambda)\,\alpha z}{n}\right]^{n-k} f_6(z,n,k). \tag{2.13}$$

Then there exist $n_2(R)$, $F_4(R)$, $F_5(R)$, *and* $F_6(R,k)$ *such that for* $|z| \leqslant R$ *and* $n \geqslant n_2(R)$,

$$|f_\nu(z,n)| \leqslant F_\nu(R) < \infty,\ \nu = 4, 5, \qquad |f_6(z,n,k)| \leqslant F_6(R,k) < \infty. \tag{2.14}$$

Moreover, if $0 < q < 1$, *then there exists a* $K_2(R,k,q)$ *such that*

$$\left\|\left[-x_1\left(\frac{z}{n}\right)\right]^{n-k}\right| \leqslant \frac{K_2(R,k,q)}{n^{q(n-k)}} \quad \text{for} \quad |z| \leqslant R \quad \text{and} \quad n \geqslant n_2(R). \tag{2.15}$$

Proof. Let $n_2(R)$ be chosen as in Lemma 2.1. By (1.11), $-x_1(z/n) = 1 + \beta z/n + x_2(z/n)$. Applying (2.1) to this we obtain (2.11) with

$$f_4(z,n) = -(-1)^\lambda \alpha(\alpha+\beta)^{2-\lambda} z^2 - f_1(z,n)/n. \tag{2.16}$$

From this and the first inequality in (2.4) it follows that there exists an $F_4(R)$ such that $|f_4(z,n)| \leqslant F_4(R) < \infty$ for $|z| \leqslant R$ and $n \geqslant n_2(R)$.

From (2.1) and (2.11) we see that if $\hat{f}(z,n)$ is defined by

$$x_1(z/(n+1)) - x_2(z/(n+1)) = 1 + \beta z/n + 2\alpha z^\lambda/n^\lambda + \hat{f}(z,n)/n^2,$$

then there exists an $\hat{F}(R)$ such that $|\hat{f}(z,n)| \leqslant \hat{F}(R) < \infty$ for $|z| \leqslant R$ and $n \geqslant n_2(R)$. The assertion regarding (2.12) follows.

From (2.11) and (2.13) it is readily seen that

$$f_6(z,n,k) = \left[1 - \frac{f_4(z,n)}{(2-\lambda)\alpha z n}\right]^{n-k}.$$

It follows from this and from (1.11), (2.1), and (2.16) that there exists an $F_6(R,k)$ such that $|f_6(z,n,k)| \leqslant F_6(R,k) < \infty$ for $|z| \leqslant R$ and $n \geqslant n_2(R)$. Finally the assertion about (2.15) is a direct consequence of (2.13) and (2.14). ∎

We note that the equalities in (1.20a) follow from Lemma 2.2 and the two equalities in (1.20b) follow from Lemma 2.1.

3. Bounds for $\{B_k(z/(n+1))\}$

The main result of this section is

THEOREM 3.1. (A) *If there exist constants* $E > 0$, $0 < \tau \leqslant 1$, $0 < \sigma \leqslant 1$ *such that*

$$\sum_{k=1}^{\infty} |\alpha_k - \alpha| < \infty \qquad \text{or} \qquad |\alpha_k - \alpha| \leqslant Ek^{-\tau} \quad \text{for} \quad k \geqslant 1 \tag{3.1a}$$

and

$$\sum_{k=1}^{\infty} |\beta_k - \beta| < \infty \quad \text{or} \quad |\beta_k - \beta| \leqslant Ek^{-\sigma} \quad \textit{for} \quad k \geqslant 1, \tag{3.1b}$$

then for each $R > 0$ *there exists a constant* $M(R)$ *such that*

$$\left| B_k\left(\frac{z}{n+1}\right) \right| \leqslant M(R) \quad \text{for} \quad |z| \leqslant R, 0 \leqslant k < n \quad \text{and} \quad n \geqslant 0. \tag{3.2}$$

(B) *If* $\lambda = 2$, *then condition* (3.1a) *can be replaced by*

$$|\alpha_k| \leqslant Ek^{\mu}, \qquad E > 0, \mu < 1, k \geqslant 1. \tag{3.3}$$

Our proof of Theorem 3.1 makes use of several lemmas. Lemma 3.2 corresponds to the comparison equation set up in [20, (2.10)]. In a disguised form (3.6) occurs in [6]. See also [15, 18]. It is stated here for completeness.

LEMMA 3.2. *Let* (a, b) *be a given pair of complex numbers, let* $\{a_k\}$ *and* $\{b_k\}$ *be given sequences of complex numbers, and let* $\{B_k\}$, $\{\delta_k\}$ *and* $\{\eta_k\}$ *be defined by*

$$B_{-1} := 0, B_0 := 1, B_k := b_k B_{k-1} + a_k B_{k-2}, \qquad k = 1, 2, 3, ..., \tag{3.4}$$

$$\delta_k = a_k - a \quad \text{and} \quad \eta_k = b_k - b, \qquad k = 1, 2, 3, ... \tag{3.5}$$

If x_1 *and* x_2 *denote the roots of the quadratic equation* $w^2 + bw - a = 0$, *then for* $n \geqslant 1$,

$$\begin{aligned}(x_1 - x_2) B_n = (-x_2)^{n+1} - (-x_1)^{n+1} \\ + \sum_{k=0}^{n-1} [(-x_2)^{n-k} - (-x_1)^{n-k}] \eta_{k+1} B_k \\ + \sum_{k=0}^{n-1} [(-x_2)^{n-k} - (-x_1)^{n-k}] \delta_{k+1} B_{k-1}.\end{aligned} \tag{3.6}$$

Returning now to the denominators $B_n(z)$ of the continued fraction (1.7) which are defined by the recurrence relations (1.19), we introduce the notation

$$B_n(z) = B_n(\alpha_1, ..., \alpha_n; \beta_1, ..., \beta_n; z) \tag{3.7}$$

which exhibits the explicit dependence of B_n on $\alpha_1, ..., \alpha_n$ and $\beta_1, ..., \beta_n$. The nth denominators $\hat{B}_n(z)$ of the related continued fraction

$$\mathop{K}_{n=1}^{\infty} \left(\frac{|\alpha_n| z^{\lambda}}{1 + |\beta_n| z} \right), \tag{3.8}$$

which are defined by the recurrence relations

$$\hat{B}_{-1}(z) := 0,\ \hat{B}_0(z) := 1,\ \hat{B}_n(z) := (1 + |\beta_n|\, z)\, \hat{B}_{n-1}(z) + |\alpha_n|\, z^{\lambda} \hat{B}_{n-2}(z),$$
$$n = 1, 2, 3, \ldots, \tag{3.9}$$

can then be expressed as follows:

$$\hat{B}_n(z) = B_n(|\alpha_1|, \ldots, |\alpha_n|; |\beta_1|, \ldots, |\beta_n|; z). \tag{3.10}$$

With α and β being given complex numbers, we also introduce the notation

$$A := |\alpha|,\ B := |\beta|,\ \delta_n := \alpha_n - \alpha,\ \eta_n = \beta_n - \beta,$$
$$\hat{\delta}_n = |\alpha_n| - A,\ \hat{\eta}_n = |\beta_n| - B, \tag{3.11}$$

and write

$$-\hat{x}_m(z) := \frac{1 + Bz}{2}\left[1 + (-1)^m \sqrt{1 + \frac{4Az^{\lambda}}{(1 + Bz)^2}}\right], \qquad m = 1, 2, \tag{3.12}$$

where $\sqrt{}$ is chosen so that Re $\sqrt{} > 0$ for $z \in \hat{D}$ and hence

$$\left|\frac{\hat{x}_1(z)}{\hat{x}_2(z)}\right| < 1 \qquad \text{for} \quad z \in \hat{D}. \tag{3.13}$$

The expression involving $\sqrt{}$ in (3.12) has two branch points $z_{\pm}$ which either are both real and negative or else are complex conjugates which lie in the half-plane $\mathrm{Re}(z) \leqslant 0$. Thus the region $\hat{D}$ in (3.13) can be (and is) chosen to contain the half-plane $\mathrm{Re}(z) > 0$ as well as $z = 0$. Therefore $r/(n+1) \in \hat{D}$ provided $r > 0$ and $n \geqslant 0$.

Applying Lemma 3.2 to $\{\hat{B}_n(z)\}$ and replacing z by $r/(n+1)$ yields

LEMMA 3.3. *For $r > 0$ and $n \geqslant 0$,*

$$\left[\hat{x}_1\left(\frac{r}{n+1}\right) - \hat{x}_2\left(\frac{r}{n+1}\right)\right] \hat{B}_n\left(\frac{r}{n+1}\right)$$
$$= \left[-\hat{x}_2\left(\frac{r}{n+1}\right)\right]^{n+1} - \left[-\hat{x}_1\left(\frac{r}{n+1}\right)\right]^{n+1}$$
$$+ \sum_{k=0}^{n-1} \left[\left(-x_2\left(\frac{r}{n+1}\right)\right)^{n-k} - \left(-x_1\left(\frac{r}{n+1}\right)\right)^{n-k}\right] \tag{3.14}$$
$$\times \left[\hat{\eta}_{k+1}\left(\frac{r}{n+1}\right) \hat{B}_k\left(\frac{r}{n+1}\right) + \hat{\delta}_{k+1}\left(\frac{r}{n+1}\right)^{\lambda} \hat{B}_{k-1}\left(\frac{r}{n+1}\right)\right].$$

LEMMA 3.4. *For each $R>0$ there exist constants $\xi_1(R)$ and $\xi_2(R)$ such that, for $0<r\leqslant R$ and $n\geqslant 0$,*

$$\left|-\hat{x}_m\left(\frac{r}{n+1}\right)\right|^{n+1}\leqslant \xi_m(R), \qquad m=1,2, \tag{3.15}$$

$$\left|\left(-\hat{x}_2\left(\frac{r}{n+1}\right)\right)^{n-k}-\left(-\hat{x}_1\left(\frac{r}{n+1}\right)\right)^{n-k}\right|\leqslant 2\xi_2(R), \quad 0\leqslant k<n, \tag{3.16}$$

$$\left|\hat{x}_1\left(\frac{r}{n+1}\right)-\hat{x}_2\left(\frac{r}{n+1}\right)\right|\geqslant 1. \tag{3.17}$$

Proof. Inequalities (3.15) are readily established by applying Lemma 2.1 (for $m=2$) and Lemma 2.2 (for $m=1$). By (3.12) with $m=2$ and (3.13) we obtain

$$\begin{aligned}&\left|\left(-\hat{x}_2\left(\frac{r}{n+1}\right)\right)^{n-k}-\left(-\hat{x}_1\left(\frac{r}{n+1}\right)^{n-k}\right)\right|\\&\leqslant\left|-\hat{x}_2\left(\frac{r}{n+1}\right)\right|^{n-k}\left[1+\left|\hat{x}_1\left(\frac{r}{n+1}\right)\Big/\hat{x}_2\left(\frac{r}{n+1}\right)\right|^{n-k}\right]\\&\leqslant 2\left|\hat{x}_2\left(\frac{r}{n+1}\right)\right|^{n-k}\leqslant 2\left|\hat{x}_2\left(\frac{r}{n+1}\right)\right|^{n+1}\leqslant 2\xi_2(R)\end{aligned}$$

for all $0<r\leqslant R, 0\leqslant k<n, n\geqslant 0$. Finally (3.17) is an immediate consequence of (3.12) ∎

The following lemma, established in [15, Lemma 2.2] and [20, p. 440] is the discrete version of Gronwall's inequality (see, e.g., [1, p. 455]). It is stated here for completeness.

LEMMA 3.5. *If a sequence $\{S_n\}$ satisfies*

$$0<S_n\leqslant K+\sum_{k=0}^{n-1}\gamma_{k+1}S_k,\ S_0\geqslant K>0, \gamma_{k+1}\geqslant 0, \qquad \text{for} \quad n\geqslant 1, \tag{3.18}$$

then

$$S_n\leqslant K\prod_{k=1}^{n}(1+\gamma_k), \qquad n=1,2,3,\dots. \tag{3.19}$$

Proof of Theorem 3.1. (A). By Lemmas 3.3 and 3.4, for $0<r\leqslant R$,

$$\begin{aligned}\hat{B}_n\left(\frac{r}{n+1}\right)\leqslant \xi_2(R)+\xi_1(R)+2\xi_2(R)\sum_{k=0}^{n-1}\Bigg[&|\hat{\eta}_{k+1}|\frac{r}{n+1}\hat{B}_k\left(\frac{r}{n+1}\right)\\&+|\hat{\delta}_{k+1}|\left(\frac{r}{n+1}\right)^{\lambda}\hat{B}_{k-1}\left(\frac{r}{n+1}\right)\Bigg]\end{aligned}$$

and since, $\hat{B}_k(r/(n+1)) \leqslant \hat{B}_k(r/(k+1))$ and $\hat{B}_{k-1}(r/(n+1)) \leqslant \hat{B}_{k-1}(r/k)$, we have

$$\hat{B}_n\left(\frac{r}{n+1}\right) \leqslant \xi_2(R) + \xi_1(R) + 2\xi_2(R) \sum_{k=0}^{n-1} \left[|\hat{\eta}_{k+1}| \left(\frac{R}{k+1}\right) \hat{B}_k\left(\frac{r}{k+1}\right) + |\hat{\delta}_{k+1}| \left(\frac{R}{k}\right)^{\lambda} \hat{B}_{k-1}\left(\frac{r}{k}\right)\right]. \tag{3.20}$$

Noting that $\hat{B}_{-1}(r/k) = 0$ and adding $2\xi_2(R)|\hat{\delta}_{n+1}|(R/n)^{\lambda} B_{n-1}(r/n)$ to the right side of (3.20) gives

$$\hat{B}_n\left(\frac{r}{n+1}\right) \leqslant \xi_1(R) + \xi_2(R) + 2\xi_2(R) \times \sum_{k=0}^{n-1} \left[|\hat{\eta}_{k+1}| \frac{R}{k+1} + |\hat{\delta}_{k+2}| \left(\frac{R}{k+1}\right)^{\lambda} \right] \hat{B}_k\left(\frac{r}{k+1}\right). \tag{3.21}$$

By setting, for $0 \leqslant k \leqslant n-1$,

$$K := \xi_1(R) + \xi_2(R), \; S_k := \hat{B}_k\left(\frac{r}{k+1}\right)$$

and

$$\gamma_{k+1} := 2\xi_2(R)\left[|\hat{\eta}_{k+1}| \frac{R}{k+1} + |\hat{\delta}_{k+2}| \left(\frac{R}{k+1}\right)^{\lambda} \right]$$

we see that $S_0 = 1 < K$, $S_k > 0$, and $\gamma_{k+1} \geqslant 0$ so that an application of Lemma 3.5 gives

$$\hat{B}_n\left(\frac{r}{n+1}\right) \leqslant (\xi_1(R) + \xi_2(R)) \prod_{k=1}^{n} \left[1 + 2\xi_2(R)\left(|\hat{\eta}_k| \frac{R}{k} + |\hat{\delta}_{k+1}| \left(\frac{R}{k}\right)^{\lambda} \right)\right]. \tag{3.22}$$

The righthand side of (3.22) forms a non-decreasing sequence which converges as $n \to \infty$ to a finite limit $M(R)$ provided that

$$\sum_{k=1}^{\infty} |\hat{\delta}_k| < \infty \quad \text{or} \quad |\hat{\delta}_k| \leqslant Ek^{-\tau}, \quad E > 0, 0 < \tau \leqslant 1, k \geqslant 1, \tag{3.23a}$$

and

$$\sum_{k=1}^{\infty} |\hat{\eta}_k| < \infty \quad \text{or} \quad |\hat{\eta}_k| \leqslant Ek^{-\sigma}, \quad E > 0, 0 < \sigma \leqslant 1, k \geqslant 1 \tag{3.23b}$$

or

$$\lambda = 2, \quad \text{(3.23b) holds,} \quad \text{and} \quad |\hat{\delta}_k| \leqslant Ek^{\mu}, E > 0, \mu < 1, k \geqslant 1. \tag{3.24}$$

Our proof is completed by observing that, since

$$|\alpha_k - \alpha| =: |\delta_k| \geqslant |\hat{\delta}_k| = ||\alpha_k| - A|$$

and

$$|\beta_k - \beta| =: |\eta_k| \geqslant |\hat{\eta}_k| = ||\beta_k| - B|,$$

the hypotheses of Theorem 3.1 imply that either (3.23) or (3.24) hold and hence

$$\left|B_k\left(\frac{z}{n+1}\right)\right| \leqslant \hat{B}_k\left(\frac{|z|}{n+1}\right) \leqslant \hat{B}_k\left(\frac{|z|}{k+1}\right) \leqslant M(R)$$
$$\text{for} \quad |z| \leqslant R, 0 \leqslant k < n, n \geqslant 0. \quad \blacksquare$$

4. Convergence of $\{B_n(z/(n+1))\}$

The main results of this paper are summarized in Theorems 4.1 and 4.2. Sufficient conditions are given for the convergence of $\{B_n(z/(n+1))\}$ and of the coefficient sequences $\{p_k^{(n)}\}_{n=0}^{\infty}$, where $B_n(z)$ denotes the nth denominator of a continued fraction of the form

$$\mathop{K}_{n=1}^{\infty}\left(\frac{\alpha_n z^\lambda}{1+\beta_n z}\right), \qquad 0 \neq \alpha_n \in \mathbb{C}, \beta_n \in \mathbb{C}, \lambda = 1 \text{ or } 2 \tag{4.1}$$

and

$$B_n\left(\frac{z}{n+1}\right) =: \sum_{k=0}^{d_n} p_k^{(n)} z^k, \qquad d_n \leqslant n. \tag{4.2}$$

The four different sets of sufficient conditions are

$$\sum_{k=1}^{\infty} |\alpha_k - \alpha| < \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} |\beta_k - \beta| < \infty; \tag{4.3}$$

$$|\alpha_k - \alpha| \leqslant Ek^{-1} \qquad \text{and} \qquad |\beta_k - \beta| \leqslant Ek^{-1}, \qquad \text{for} \quad E > 0, k \geqslant 1; \tag{4.4}$$

$$|\alpha_k - \alpha| \leqslant Ek^{-\tau} \qquad \text{and} \qquad |\beta_k - \beta| \leqslant Ek^{-\sigma}, \qquad E > 0, k \geqslant 1, \tag{4.5a}$$

where

$$0 < \tau \leqslant 1 \quad \text{and} \quad 0 < \sigma < 1 \quad \text{or} \quad 0 < \tau < 1 \quad \text{and} \quad 0 < \sigma \leqslant 1; \tag{4.5b}$$

$$\lambda = 2, |\alpha_k| \leqslant Ek^{\tau} \qquad \text{and} \qquad |\beta_k - \beta| \leqslant Ek^{-\sigma}, \qquad \text{for} \quad E > 0, k \geqslant 1, \tag{4.6a}$$

where

$$0<\tau<1 \quad \text{and} \quad 0<\sigma\leqslant 1. \tag{4.6b}$$

For convenience we recall the notation

$$\gamma := (2-\lambda)\,\alpha+\beta, \qquad \Gamma := \tfrac{1}{2}[(-1)^{\lambda}\, 2\alpha(\alpha+\beta)^{2-\lambda}-\gamma^2],$$
$$\Delta := \sum_{k=2}^{\infty}(\alpha_k-\alpha), \qquad \Theta := \sum_{k=1}^{\infty}(\beta_k-\beta), \tag{4.7}$$

when the series in question are convergent.

THEOREM 4.1. (A) *If* (4.3) *holds and* $t_1(z,n)$ *is defined by*

$$B_n\left(\frac{z}{n+1}\right)-e^{\gamma z}=e^{\gamma z}\left[\frac{\Gamma z^2+(\Theta-\beta)\,z}{n}+\frac{(\Delta-2\alpha)\,z^{\lambda}}{n^{\lambda}}\right]+\frac{t_1(z,n)}{n}, \tag{4.8}$$

then there exists a function $T_1(R,n)$ *such that, for* $|z|\leqslant R$ *and* $n\geqslant 1$,

$$|t_1(z,n)|\leqslant T_1(R,n) \quad \text{and} \quad \lim_{n\to\infty} T_1(R,n)=0. \tag{4.9}$$

(B) *Let* $t_v(z,n)$, $v=2,3,4$ *be defined by*

$$B_n\left(\frac{z}{n+1}\right)-e^{\gamma z}=t_2(z,n)\,\frac{\log n}{n}, \tag{4.10}$$

$$B_n\left(\frac{z}{n+1}\right)-e^{\gamma z}=\frac{t_3(z,n)}{n^{\min(\sigma,\tau)}}, \tag{4.11}$$

$$B_n\left(\frac{z}{n+1}\right)-e^{\gamma z}=\frac{t_4(z,n)}{n^{\min(\sigma,\,1-\tau)}}. \tag{4.12}$$

Then there exist functions $T_v(R)$ *such that for* $v=2,3,4$, *if* $(4.2+v)$ *holds then*

$$|t_v(z,n)|\leqslant T_v(R) \quad \textit{for} \quad |z|\leqslant R \quad \text{and} \quad n\geqslant 1. \tag{4.13}$$

THEOREM 4.2. *Let* $p_k^{(n)}$ *be defined by* (4.2).

(A) *If* (4.3) *holds then, for* $n\geqslant 1$,

$$p_0^{(n)}=1, \qquad p_1^{(n)}=\gamma+\frac{(2-\lambda)(\Delta-2\alpha)+(\Theta-\beta)}{n}+o\left(\frac{1}{n}\right), \tag{4.14a}$$

$$p_k^{(n)}=\frac{\gamma^k}{k!}+\frac{\gamma^{k-2}[(k-1)\,\Gamma+\gamma(\Theta-\beta)]}{n(k-1)!}$$
$$+\frac{\gamma^{k-\lambda}(\Delta-2\alpha)}{n^{\lambda}(k-\lambda)!}+o\left(\frac{1}{n}\right), \qquad 2\leqslant k\leqslant d_n. \tag{4.14b}$$

(B) *Moreover, for* $k = 1, 2, \ldots, d_n$ *and* $n \geqslant 1$,

$$p_0^{(n)} = 1 \quad \text{and} \quad p_k^{(n)} = \frac{\gamma^k}{k!} + \begin{cases} O\left(\dfrac{\log n}{n}\right) & \text{if (4.4) holds,} \\ O\left(\dfrac{1}{n^{\min(\sigma, \tau)}}\right) & \text{if (4.5) holds,} \\ O\left(\dfrac{1}{n^{\min(\sigma, 1-\tau)}}\right) & \text{if (4.6) holds.} \end{cases} \tag{4.15}$$

Proof of Theorem 4.2. It follows from the recurrence relations (1.19) that $p_0^{(n)} = B_n(0) = 1$ for all $n \geqslant 0$. By the Cauchy integral formulas one has for $k = 1, 2, \ldots, d_n$, and $n \geqslant 1$,

$$p_k^{(n)} - \frac{\gamma^k}{k!} = \frac{1}{2\pi i} \oint_{|z| = r} \frac{B_n(z/(n+1)) - e^{\gamma z}}{z^{k+1}} \, dz. \tag{4.16}$$

The assertions of Theorem 4.2 follow from this and from the estimates given for $B_n(z/(n+1)) - e^{\gamma z}$ in Theorem 4.1. ∎

The following lemma is used in our proof of Theorem 4.1. It is also convenient to introduce the notation

$$\sigma_{n,k}(z) := \left[-x_2\left(\frac{z}{n+1}\right)\right]^{n-k} - \left[-x_1\left(\frac{z}{n+1}\right)\right]^{n-k}, \tag{4.17}$$

and

$$\begin{aligned} X_{n,k}(z) &:= \sigma_{n,k}(z)\, \delta_{k+1} B_{k-1}\left(\frac{z}{n+1}\right), \\ Y_{n,k}(z) &:= \sigma_{n,k}(z)\, \eta_{k+1} B_k\left(\frac{z}{n+1}\right), \end{aligned} \tag{4.18}$$

and to recall $\delta_k := \alpha_k - \alpha$, $\eta_k := \beta_k - \beta$, $\gamma := (2 - \lambda)\alpha + \beta$.

Lemma 4.3. (A) *If* $\sum_{k=1}^{\infty} |\delta_k| < \infty$ *and* $s_1(z, n)$ *is defined by*

$$s_1(z, n) := \sum_{k=0}^{n-1} X_{n,k}(z) - e^{\gamma z} \sum_{k=2}^{\infty} \delta_k, \tag{4.19}$$

then there exists a sequence of functions $\{S_1(R, n)\}$ *satisfying for all* $|z| \leqslant R$ *and* $n \geqslant 1$

$$|s_1(z, n)| \leqslant S_1(R, n) \quad \text{and} \quad \lim_{n \to \infty} S_1(R, n) = 0. \tag{4.20}$$

(B) *If* $\sum_{k=1}^{\infty} |\eta_k| < \infty$ *and* $s_2(z, n)$ *is defined by*

$$s_2(z, n) := \sum_{k=0}^{n-1} Y_{n,k}(z) - e^{\gamma z} \sum_{k=1}^{\infty} \eta_k, \tag{4.21}$$

then there exists a sequence of functions $\{S_2(R, n)\}$ *satisfying for all* $|z| \leqslant R$ *and* $n \geqslant 1$

$$|s_2(z, n)| \leqslant S_1(R, n) \qquad \text{and} \qquad \lim_{n \to \infty} S_2(R, n) = 0. \tag{4.22}$$

Proof of Lemma 4.3. By (1.12) and (1.13)

$$\left|\frac{-x_1(z/(n+1))}{-x_2(z/(n+1))}\right| < 1 \qquad \text{for all} \quad |z| \leqslant R \qquad \text{and} \qquad n \geqslant n_0(R).$$

By a proof similar to that used for (3.15) with $m = 2$ one can show that for each $R > 0$ there exists a constant $\xi_2(R)$ satisfying

$$\left|-x_2\left(\frac{z}{n+1}\right)\right|^{n-k} \leqslant \left|-x_2\left(\frac{z}{n+1}\right)\right|^{n+1}$$
$$\leqslant \xi_2(R) \qquad \text{for} \quad |z| \leqslant R, 0 \leqslant k < n, n \geqslant n_0(R).$$

Combining these results with (4.17) yields

$$|\sigma_{n,k}(z)| = \left|\left(-x_2\left(\frac{z}{n+1}\right)\right)^{n-k}\right|$$
$$\times \left|1 + \left(\frac{-x_1(z/(n+1))}{-x_2(z/(n+1))}\right)^{n-k}\right| \leqslant 2\xi_2(R) \tag{4.23}$$

for $|z| \leqslant R$, $0 \leqslant k < n$, $n \geqslant n_0(R)$. By (2.5), (2.15), and (4.17), for each $R > 0$ there exist $K_3(R, k)$ and $n_2(R) \geqslant n_0(R)$ such that

$$|\sigma_{n,k}(z) - e^{\gamma z}| \leqslant \frac{K_3(R, k)}{n} \qquad \text{for} \quad |z| \leqslant R, 0 \leqslant k < n, n \geqslant n_2(R). \tag{4.24}$$

Since $B_n(0) = 1$, for each $R > 0$ there exists $K_4(R, k)$ such that

$$\left|B_k\left(\frac{z}{n+1}\right) - 1\right| \leqslant \frac{K_4(R, k)}{n} \qquad \text{for} \quad |z| \leqslant R, 0 \leqslant k < n, n \geqslant 1. \tag{4.25}$$

(A) Suppose $n > 2m$. Then we can write

$$\sum_{k=0}^{n-1} X_{n,k}(z) - e^{\gamma z} \sum_{k=1}^{\infty} \delta_{k+1}$$
$$= \sum_{k=1}^{m} [\sigma_{n,k}(z) - e^{\gamma z}]\, \delta_{k+1} B_{k-1}\left(\frac{z}{n+1}\right)$$
$$+ \sum_{k=1}^{m} e^{\gamma z}\delta_{k+1}\left[B_{k-1}\left(\frac{z}{n+1}\right) - 1\right]$$
$$+ \sum_{k=m+1}^{n-1} X_{n,k}(z) - \sum_{k=m+1}^{\infty} e^{\gamma z}\delta_{k+1}. \tag{4.26}$$

Let $\varepsilon > 0$ and $R > 0$ be given. Then by (4.23), (4.24), and (4.25) there exists an m so large that the third and fourth sums on the right side of (4.32) are each in absolute value less than $\varepsilon/4$ for $|z| \leqslant R$ and $n \geqslant n_3(R) = \max(m, n_2(R))$. Now holding m fixed we choose $n_4(R) \geqslant n_3(R)$ so large that

$$\max_{0 \leqslant k \leqslant m} \left| B_{k-1}\left(\frac{z}{n+1}\right) - 1 \right| < \frac{\varepsilon}{4 \sum_{k=1}^{m} |e^{\gamma z}\delta_{k+1}|}, \qquad \text{for } |z| \leqslant R,\, n \geqslant n_4(R),$$

and

$$\max_{0 \leqslant k \leqslant m} |\sigma_{n,k}(z) - e^{\gamma z}| < \frac{\varepsilon}{4M(R) \sum_{k=1}^{m} |\delta_{k+1}|}, \qquad \text{for } |z| \leqslant R,\, n \geqslant n_4(R),$$

where $M(R)$ satisfies (3.2) of Theorem 3.1. Combining these results with (4.26) establishes (A). A proof of (B) can be given that is completely analogous to that given for (A); hence it is omitted. ▌

Proof of Theorem 4.1. Let $R > 0$ be given. An application of Lemma 3.2 to $B_n(z)$ with z replaced by $z/(n+1)$ yields, for $|z| \leqslant R$ and $n \geqslant n_0(R)$,

$$B_n\left(\frac{z}{n+1}\right) = \left[x_1\left(\frac{z}{n+1}\right) - x_2\left(\frac{z}{n+1}\right)\right]^{-1}$$
$$\times \left\{\left[-x_2\left(\frac{z}{n+1}\right)\right]^{n+1} - \left[-x_1\left(\frac{z}{n+1}\right)\right]^{n+1}\right.$$
$$\left. + \frac{z^\lambda}{(n+1)^\lambda} \sum_{k=0}^{n-1} X_{n,k}(z) + \frac{z}{n+1} \sum_{k=0}^{n-1} Y_{n,k}(z)\right\}. \tag{4.27}$$

If $h_1(z, n)$ is defined by

$$\left[-x_1\left(\frac{z}{n+1}\right)\right]^{n+1} = \frac{h_1(z, h)}{n^2}, \tag{4.28a}$$

then by (2.15) there exist an $H_1(R)$ and $n_2(R)$ such that

$$|h_1(z,n)| \leqslant H_1(R) < \infty \qquad \text{for} \quad |z| \leqslant R, n \geqslant n_2(R). \tag{4.28b}$$

Substitution of (2.12), (2.3), and (4.28a) into (4.27) gives

$$B_n\left(\frac{z}{n+1}\right) = \left[1 - \frac{\beta z}{n} - \frac{2\alpha z^\lambda}{n^\lambda} + \frac{f_5(z,n)}{n^2}\right]\left\{e^{\gamma z} + \frac{\Gamma z^2}{n} e^{\gamma z} + \frac{f_3(z,n+1,0)}{n^2}\right.$$
$$\left. + \frac{h_1(z,n)}{n^2} + \left(\frac{z}{n+1}\right)^\lambda \sum_{k=0}^{n-1} X_{n,k}(z) + \frac{z}{n+1}\sum_{k=0}^{n-1} Y_{n,k}(z)\right\}. \tag{4.29}$$

(A) If (4.3) holds, then we can substitute (4.19) and (4.21) into (4.29) and, after multiplying and rearranging terms and applying Lemmas 2.1, 2.2, and 4.3, we conclude that assertion (A) holds.

Before proving (B) we note that by Theorem 3.1, (4.18), and (4.23) one has for $|z| \leqslant R$ and $n \geqslant n_0(R)$

$$\left|\sum_{k=0}^{n-1} X_{n,k}(z)\right| \leqslant 2\xi_2(R)\, M(R) \sum_{k=1}^{n} |\delta_k|,$$
$$\left|\sum_{k=0}^{n-1} Y_{n,k}(z)\right| \leqslant 2\xi_2(R)\, M(R) \sum_{k=1}^{n} |\eta_k|. \tag{4.30}$$

(B) If (4.4) holds, then it can be seen that

$$\sum_{k=1}^{n} |\delta_k| = O(\log n) \qquad \text{and} \qquad \sum_{k=1}^{n} |\eta_k| = O(\log n). \tag{4.31}$$

If (4.5) holds, then

$$\sum_{k=1}^{n} |\delta_k| = O(n^{1-\tau}) \qquad \text{or} \qquad \sum_{k=1}^{n} |\eta_k| = O(n^{1-\sigma}).$$

If (4.6) holds then

$$\sum_{k=1}^{n} |\delta_k| = O(n^{\tau+1}) \qquad \text{and} \qquad \sum_{k=1}^{n} |\eta_k| = O(n^{1-\sigma}), \qquad 0 < \sigma < 1.$$

We then deduce assertion (B) by applying (4.28b), (4.30), (4.31), and Lemmas 2.1 and 2.2 to (4.29).

By means of the final theorem in this section one can see that the results (Theorems 4.1 and 4.2) for denominators of continued fractions (4.1) are all applicable to the nth numerators $A_n(z)$, as well.

THEOREM 4.4. *Let* $A_n(z)$ *denote the* n*th numerator of the continued fraction* (4.1) *and let* $\{B_n^\dagger(z)\}$ *be defined by* $B_n^\dagger(z) := A_{n+1}(z)/\alpha_1 z^\lambda$, $n \geqslant -1$. *Then* $B_n^\dagger(z)$ *is the* n*th denominator of the continued fraction*

$$\underset{n=1}{\overset{\infty}{K}} \left(\frac{\alpha_{n+1} z^\lambda}{1+\beta_{n+1} z}\right), \tag{4.32}$$

$\{A_n(z)\}$ *and* $\{B_n^\dagger(z)\}$ *converge and diverge together, and*

$$\lim_{n\to\infty} \frac{1}{\alpha_1}\left(\frac{n+1}{z}\right)^\lambda A_n\left(\frac{z}{n+1}\right) = \lim_{n\to\infty} B_{n-1}^\dagger\left(\frac{z}{n+1}\right) \tag{4.33}$$

if the limit on the right side exists.

Proof. It follows from the recurrence relations defining $\{A_n(z)\}$ that $\{B_n^\dagger(z)\}$ satisfies

$$B_{-1}^\dagger(z) = 0,\ B_0^\dagger(z) = 1,\ B_n^\dagger(z) = (1+\beta_{n+1} z)\, B_{n-1}^\dagger(z) + \alpha_{n+1} z^\lambda B_{n-2}^\dagger(z),$$
$$n = 1, 2, 3, \ldots .$$

Hence $B_n^\dagger(z)$ is the nth denominator of (4.32). ∎

5. APPLICATIONS AND EXAMPLES.

In this section we discuss briefly some important types of continued fractions that are subsumed under (1.7) and describe interpretations of Theorems 4.1 and 4.2 in these special situations. Most of the discussion is restricted to the (A) parts of these theorems, since the results for the (B) parts do not substantially differ from those given by Theorems 4.1 and 4.2. We also establish connections with orthogonal polynomial sequences.

5.1. REGULAR C-FRACTIONS. Let $B_n(z)$ denote the nth denominator of a *regular C-fraction*

$$\underset{n=1}{\overset{\infty}{K}} \left(\frac{a_n z}{1}\right), \qquad 0 \neq a_n \in \mathbb{C}, \qquad \lim_{n\to\infty} a_n = a \in \mathbb{C}. \tag{5.1}$$

In the terminology of Section 4 we have $\alpha_n = a_n$, $\alpha = a$, $\beta_n = \beta = 0$, $\lambda = 1$, $\gamma = a$, $\Gamma = -\frac{3}{2} a^2$, $\Delta := \sum_{k=2}^\infty (a_k - a)$, $\Theta = 0$, and $p_k^{(n)}$ is defined by (4.2). *If*

$$\sum_{k=1}^\infty |a_k - a| < \infty \tag{5.2}$$

and $t_1(z, n)$ *is defined by*

$$B_n\left(\frac{z}{n+1}\right)-e^{az}=e^{az}\left[\frac{(\Delta-2a)\,z-(3/2)\,a^2z^2}{n}\right]+\frac{t_1(z,n)}{n}, \tag{5.3}$$

then there exists a function $T_1(R, n)$ *such that*

$$|t_1(z,n)|\leqslant T_1(R,n) \quad \text{for} \quad |z|\leqslant R,\, n\geqslant 1 \quad \text{and} \quad \lim_{n\to\infty} T_1(R,n)=0. \tag{5.4}$$

Moreover, for $n\geqslant 1$,

$$p_0^{(n)}=1 \quad \textit{and} \quad p_k^{(n)}=\frac{a^k}{k!}+\frac{a^{k-1}[\Delta-\frac{1}{2}a(3k+1)]}{n(k-1)!}+o\left(\frac{1}{n}\right), \tag{5.5}$$

for $1\leqslant k\leqslant d_n$, *where* $d_{2m}=d_{2m+1}=m$ *for* $m\geqslant 1$.

It is known that the sequence of approximants $\{A_n(z)/B_n(z)\}$ of (5.1) forms a "staircase" in the corresponding Padé table [9, Theorem 5.19]. For the regular C-fraction (5.1) of Gauss, the coefficients a_n are given by

$$a_{2m+1}=-\frac{(A+m)(C-B+m)}{(C+2m)(C+2m+1)}, \qquad a_{2m}=-\frac{(B+m)(C-A+m)}{(C+2m-1)(C+2m)}, \tag{5.6}$$

where the complex constants A, B, C are chosen in such a manner that $0\neq a_n\in\mathbb{C}$ for $n\geqslant 1$. It is well known that this regular C-fraction (5.1) converges to a function $g(z)$, meromorphic in the domain $D_1:=[z: 0<\arg(z-1)<2\pi]$ and holomorphic at $z=0$ with $g(0)=0$; moreover, $g(z)$ provides the analytic continuation into D_1 of

$$\frac{{}_2F_1(A,B;C;z)}{{}_2F_1(A,B+1;C+1;z)}-1, \tag{5.7a}$$

where ${}_2F_1(A, B; C; z)$ denotes the hypergeometric series

$${}_2F_1(A,B;C;z):=\sum_{n=0}^{\infty}\frac{(A)_n(B)_n}{(C)_n}\frac{z^n}{n!}, \tag{5.7b}$$

convergent for $|z|<1$ and $(A)_0:=1$, $(A)_n:=A(A+1)\cdots(A+n-1)$, for $n\geqslant 1$ [9, Theorem 6.1]. It can readily be shown that

$$\lim_{n\to\infty} a_n=a=-\tfrac{1}{4} \tag{5.8}$$

and

$$r_n:=\max_{m\geqslant n}|a_m+\tfrac{1}{4}|=|1-2A+2B|\,O\left(\frac{1}{n}\right)+O\left(\frac{1}{n^2}\right) \tag{5.9}$$

[2, Eq. (5.4)]. It follows that

$$|a_n + \tfrac{1}{4}| \leqslant En^{-2}, \qquad \text{for} \quad E > 0, n \geqslant 1 \qquad \text{if} \quad A = B + \tfrac{1}{2} \tag{5.10a}$$

and

$$|a_n + \tfrac{1}{4}| \leqslant En^{-1}, \qquad \text{for} \quad E > 0, n \geqslant 1 \qquad \text{if} \quad A \neq B + \tfrac{1}{2}. \tag{5.10b}$$

Thus by Theorems 4.1 and 4.2

$$\lim_{n \to \infty} B_n\left(\frac{z}{n+1}\right) = e^{-z/4} \qquad \text{for all} \quad z \in \mathbb{C}, \tag{5.11}$$

the order of the convergence being given by the theorems of Section 4; moreover, for $1 \leqslant k \leqslant d_n$ and $n \geqslant 1$,

$$p_k^{(n)} = \frac{(-1/4)^k}{k!} + \begin{cases} \dfrac{(-1/4)^{k-1}\,[\Delta + (1/8)(3k+1)]}{n(k-1)!} + o\left(\dfrac{1}{n}\right), & \text{if } A = B + \dfrac{1}{2}, \\ O\left(\dfrac{\log n}{n}\right) & \text{if } A \neq B + \dfrac{1}{2}. \end{cases} \tag{5.12}$$

5.2. General T-Fractions. General *T-Fractions* are continued fractions of the form

$$\mathop{K}_{n=1}^{\infty}\left(\frac{F_n z}{1 + G_n z}\right), \qquad 0 \neq F_n \in \mathbb{C}, G_n \in \mathbb{C}. \tag{5.13}$$

They are distinguished by the property that, if all $G_n \neq 0$, then their sequence of approximants forms the main diagonal in a two-point Padé table [9, Sect. 7.3]. If $F_n > 0$ and $G_n > 0$ for all n, then (5.13), called a positive T-fraction, is intimately related to the strong Stieltjes moment problem [10]. The denominators of positive T-fractions give rise to orthogonal Laurent polynomials on $[0, \infty)$ [8]. Here we consider general T-fractions (5.13) satisfying

$$\lim_{n \to \infty} F_n = F \in \mathbb{C} \qquad \text{and} \qquad \lim_{n \to \infty} G_n = G \in \mathbb{C}. \tag{5.14}$$

In the terminology of Section 4 we have $\alpha_n = F_n$, $\beta_n = G_n$, $\lambda = 1$, $\alpha = F$, $\beta = G$, $\gamma = F + G$, $\Gamma = -\frac{1}{2}(F + G)(3F + G)$, $\Delta = \sum_2^\infty (F_k - F)$, $\Theta = \sum_1^\infty (G_k - G)$, if these series converge. $p_k^{(n)}$ is defined by (4.2), where $B_n(z)$ denotes the nth denominator of (5.13). *If*

$$\sum_{k=1}^{\infty} |F_k - F| < \infty \qquad \textit{and} \qquad \sum_{k=1}^{\infty} |G_k - G| < \infty, \tag{5.15}$$

and $t_1(z, n)$ *is defined by*

$$B_n\left(\frac{z}{n+1}\right) - e^{\gamma z} = e^{\gamma z}\left[\frac{\Gamma z^2 + (\Delta + \Theta - 2F - G)\, z}{n}\right] + \frac{t_1(z, n)}{n}, \quad (5.16)$$

then there exists a function $T_1(R, n)$ such that

$$|t_1(z, n)| \leqslant T_1(R, n) \quad \text{for} \quad |z| \leqslant R, n \geqslant 1 \quad \text{and} \quad \lim_{n \to \infty} T_1(R, n) = 0.$$

Moreover, for $n \geqslant 1$

$$p_0^{(n)} = 1, \qquad p_1^{(n)} = \gamma + \frac{\Delta + \Theta - 2F - G}{n} + o\left(\frac{1}{n}\right), \quad (5.17a)$$

$$p_n^{(k)} = \frac{\gamma^k}{k!} + \frac{\gamma^{k-2}[(k-1)\,\Gamma + \gamma(\Delta + \Theta - 2F - G)]}{n(k-1)!}$$

$$+ o\left(\frac{1}{n}\right), \qquad 2 \leqslant k \leqslant d_n \leqslant n. \quad (5.17b)$$

The case in which $\gamma = F + G = 0$ has received special attention by Waadeland and others (see, e.g., [18 and references therein] [4]). In particular one obtains $F = -G = 0$ when

$$F_1 := 1, F_n := \frac{B+n-1}{(C+n-2)(C+n-1)}, n \geqslant 2;$$

$$G_n := \frac{-1}{C+n-1}, n \geqslant 1, \quad (5.18)$$

where B and C are complex constants chosen so that $0 \neq F_n \in \mathbb{C}$ and $G_n \in \mathbb{C}$ for $n \geqslant 1$. With these coefficients the general T-fraction (5.13) converges to the meromorphic function

$$f(z) := \frac{{}_1F_1(B+1; C+1; z)}{{}_1F_1(B; C; z)} - 1, \qquad {}_1F_1(B; C; z) := \sum_{n=0}^{\infty} \frac{(B)_n}{(C)_n} \frac{z^n}{n!}. \quad (5.19)$$

The convergence is uniform on compact subsets of $\mathbb{C}$ containing no pole of $f(z)$. It follows from (5.18) that

$$|F_k| \leqslant Ek^{-1} \quad \text{and} \quad |G_k| \leqslant Ek^{-1}, \qquad E > 0, k \geqslant 1. \quad (5.20)$$

Hence by Theorems 4.1 and 4.2, for $z \in \mathbb{C}$, $n \geqslant 1$,

$$\lim_{n \to \infty} B_n\left(\frac{z}{n+1}\right) = e^{\gamma z} = 1, p_0^{(n)} = 1, p_k^{(n)} = O\left(\frac{\log n}{n}\right),$$

$$n \to \infty, 1 \leqslant k \leqslant d_n. \quad (5.21)$$

5.3. ASSOCIATED CONTINUED FRACTIONS AND J-FRACTIONS. Here we let $B_n(z)$ denote the nth denominator of an associated continued fraction

$$\mathop{K}_{n=1}^{\infty}\left(\frac{-K_n z^2}{1+L_n z}\right), \qquad 0 \neq K_n \in \mathbb{C},\ L_n \in \mathbb{C}. \tag{5.22}$$

Its approximants form the main diagonal of a Padé table. Using terminology from Section 4 we have $\alpha_n = -K_n$, $\beta_n = L_n$, $\lambda = 2$, $\alpha = -K = -\lim K_n$, $\beta = L = \lim L_n$, $\Delta = -\sum_2^\infty (K_n - K)$ and $\Theta = \sum_1^\infty (L_n - L)$ if the limits in question exist, $\alpha = K = 0$ if $\lim K_n$ does not exist, $\gamma = L$ and $\Gamma = -(K + \frac{1}{2}L^2)$.

If

$$\sum_{k=1}^{\infty} |K_k - K| < \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} |L_k - L| < \infty, \tag{5.23}$$

and $t_1(z, n)$ *is defined by*

$$B_n\left(\frac{z}{n+1}\right) - e^{Lz} = e^{Lz}\left[\frac{\Gamma z^2 + (\Theta - L)\, z}{n}\right] + \frac{t_1(z, n)}{n}, \tag{5.24}$$

then there exists a function $T_1(R, n)$ satisfying

$$|t_1(z, n)| \leqslant T_1(R, n) \quad \text{for} \quad |z| \leqslant R,\ n \geqslant 1 \quad \text{and} \quad \lim_{n\to\infty} T_1(R, n) = 0.$$

$$p_0^{(n)} = 1, \qquad p_1^{(n)} = L + \frac{\Theta - L}{n} + o\left(\frac{1}{n}\right), \tag{5.25a}$$

$$p_k^{(n)} = \frac{L^k}{k!} + \frac{L^{k-2}[(k-1)\,\Gamma + L(\Theta - L)]}{n(k-1)!} + o\left(\frac{1}{n}\right), \qquad 2 \leqslant k \leqslant d_n \leqslant n. \tag{5.25b}$$

Closely related to the associated continued fractions (5.22) are the *J-fractions*

$$\mathop{K}_{n=1}^{\infty}\left(\frac{-K_n}{L_n + \zeta}\right), \qquad K_1 = 1,\ 0 \neq K_n \in \mathbb{C},\ L_n \in \mathbb{C}. \tag{5.26}$$

If $B_n(z)$ and $\tilde{B}_n(\zeta)$ denote the nth denominators of (5.22) and (5.26), respectively, then it can be shown that

$$B_n(z) = z^n \tilde{B}_n\left(\frac{1}{z}\right) \qquad \text{and} \qquad \tilde{B}_n(\zeta) = \zeta^n B_n\left(\frac{1}{\zeta}\right), \qquad n \geqslant 0. \tag{5.27}$$

It follows that

$$\lim_{n\to\infty}\left(\frac{z}{n+1}\right)^n \tilde{B}_n\left(\frac{n+1}{z}\right) = \lim_{n\to\infty} B_n\left(\frac{z}{n+1}\right), \tag{5.28}$$

provided the limit on the right side exists. Thus convergence of $B_n(z/(n+1))$ can be applied to $\tilde{B}_n(\zeta)$ by means of (5.28).

Of special interest are *real J-fractions* (5.26) with the further condition

$$K_n > 0 \quad \text{for} \quad n \geqslant 2 \quad \text{and} \quad L_n \in \mathbb{R} \quad \text{for} \quad n \geqslant 1. \tag{5.29}$$

One reason for their importance is that every orthogonal (monic) polynomial sequence can be realized as the sequence of nth denominators of a real J-fraction and, conversely, the sequence $\{\tilde{B}_n(\zeta)\}$ of every real J-fraction is an orthogonal (monic) polynomial sequence with respect to some distribution function $\psi(t)$ on $\mathbb{R}$ [9, Favard's Theorem, p. 254].

As an example we consider Legendre polynomials $\{P_n(x)\}$ defined by

$$P_0(x) := 1,\ P_1(x) := x,\ P_n(x) := \frac{2n-1}{n} x P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x),$$
$$n = 2, 3, 4, \ldots, \tag{5.30}$$

It can be seen that $P_n(x)$ is the nth denominator of

$$\frac{1}{x} + \frac{-(1/2)}{(3/2)\,x} + \frac{-(2/3)}{(5/3)\,x} + \frac{-(3/4)}{(7/4)\,x} + \cdots + \frac{-((n-1)/n)}{((2n-1)/n)\,x} + \cdots, \tag{5.31}$$

which is equivalent to the real J-fraction

$$\mathop{K}_{n=1}^{\infty}\left(\frac{-K_n}{x}\right),\ K_1 := -1,\ K_n = \frac{(n-1)^2}{(2n-1)(2n-3)}, \qquad n = 2, 3, 4, \ldots. \tag{5.32}$$

If $\tilde{B}_n(x)$ denotes the nth denominator of (5.32), then it is readily seen that

$$\tilde{B}_n(x) = \prod_{k=1}^{n}\left(\frac{k}{2k-1}\right) P_n(x) = \frac{2^n (n!)^2}{(2n)!} P_n(x), \qquad n = 0, 1, 2, \ldots. \tag{5.33}$$

Thus $\tilde{B}_n(x)$ is the monic Legendre polynomial of degree n. Performing an equivalence transformation on the real J-fraction (5.32) and replacing x^{-1} by z yields the associated continued fraction

$$\frac{1}{z} \mathop{K}_{n=1}^{\infty}\left(\frac{-K_n z^2}{1}\right)$$
$$= \frac{z}{1} + \frac{-(1^2/1.3)\,z^2}{1} + \frac{-(2^2/3\cdot 5)\,z^2}{1} + \cdots + \frac{-(n-1)^2/(2n-1)(2n-3))\,z^2}{1} + \cdots. \tag{5.34}$$

Letting $B_n(z)$ denote the nth denominator of (5.34) we obtain

$$B_n(z) = z^n \tilde{B}_n\left(\frac{1}{z}\right) \quad \text{and} \quad \tilde{B}_n(x) = x^n B_n\left(\frac{1}{x}\right), \quad n = 0, 1, 2, \dots . \tag{5.35}$$

Since $-K := \lim_{n \to \infty} -K_n = -\frac{1}{4}$, $K_{n+1} - K = 1/4(4n^2 - 1)$, and hence

$$\Delta := -\sum_{n=1}^{\infty} (K_{n+1} - K) = -\tfrac{1}{8} \qquad \text{(a telescoping series)}, \tag{5.36}$$

an application of Theorem 4.1 and 4.2 with $\gamma = L = L_n = \Theta = 0$ and $\Gamma = -\frac{1}{4}$ yields

$$B_n\left(\frac{z}{n+1}\right) - 1 = (-\tfrac{1}{4}z^2/n) + (t_1(z, n)/n), \tag{5.37}$$

where there exists a function $T_1(R, n)$ satisfying

$$|t_1(z, n)| \leqslant T_1(R, n) \quad \text{for} \quad |z| \leqslant R, n \geqslant 1, \quad \text{and} \quad \lim_{n \to \infty} T_1(R, n) = 0.$$

Moreover,

$$p_0^{(n)} = 1 \quad \text{and} \quad p_k^{(n)} = o\left(\frac{1}{n}\right) \quad \text{for} \quad 1 \leqslant k \leqslant d_n, n = 1, 2, 3, \dots, \tag{5.38}$$

where $p_k^{(n)}$ is defined by (4.2). From (5.35) and (5.37) we see that

$$\lim_{n \to \infty} \left(\frac{z}{n+1}\right)^n \tilde{B}_n\left(\frac{n+1}{z}\right) = \lim_{n \to \infty} B_n\left(\frac{z}{n+1}\right) = 1, \tag{5.39}$$

the convergence being uniform on all compact subsets of $\mathbb{C}$.

Another interesting result along this line can be obtained by considering the regular C-fraction

$$\mathop{K}_{n=1}^{\infty}\left(\frac{-K_n w}{1}\right), \qquad K_n \text{ defined by (5.32)}, \tag{5.40}$$

whose nth denominator is denoted by $\hat{B}_n(w)$. We write

$$\hat{B}_n\left(\frac{w}{n+1}\right) = \sum_{k=0}^{\hat{d}_n} \hat{p}_k^{(n)} w^k, \qquad \hat{p}_{\hat{d}_n}^{(n)} \neq 0, \; w = z^2. \tag{5.41}$$

It follows that

$$\hat{B}_n(w) = \hat{B}_n(z^2) = B_n(z), \quad n = 0, 1, 2, \dots . \tag{5.42}$$

An application of Theorems 4.1 and 4.2 with $\gamma = a = -\lim K_n = -\frac{1}{4}$, $\Gamma = -\frac{3}{32}$, $\Delta = -\frac{1}{8}$ yields

$$\hat{B}_n\left(\frac{z^2}{n+1}\right) - e^{-z^2/4} = e^{-z^2/4}\left[\frac{(3/8)\,z^2 - (3/32)\,z^4}{n}\right] + \frac{\hat{t}_1(z^2, n)}{n}, \quad (5.43a)$$

where

$$|\hat{t}_1(z^2, n)| \leqslant \hat{T}_1(R^2, n) \quad \text{for} \quad |z| \leqslant R, n \geqslant 1 \quad \text{and} \quad \lim_{n\to\infty} \hat{T}_n(R^2, n) = 0. \quad (5.43b)$$

Moreover, for $1 \leqslant k \leqslant \hat{d}_n$ and $n \geqslant 1$,

$$\hat{p}_1^{(n)} = 0 \quad \text{and} \quad \hat{p}_k^{(n)} = \frac{(-1/4)^k}{k!} + \frac{(-1/4)^{k-1}\,(3/8)\,k}{n(k-1)!} + o\left(\frac{1}{n}\right). \quad (5.44)$$

From (5.42) one also sees that $d_n = 2\hat{d}_n$ and $p_k^{(n)} = 0$ if k is odd. By further use of (5.42) and (5.43) one can show that, for $1 \leqslant k \leqslant \hat{d}_n$ and $n \geqslant 1$,

$$p_{2k}^{(n)} = \frac{\hat{p}_k^{(n)}}{(n+1)^k} = \frac{1}{(n+1)^k}\left[\frac{(-1/4)^k}{k!} + \frac{(-1/4)^{k-1}\,(3/8)\,k}{n(k-1)!} + o\left(\frac{1}{n}\right)\right]. \quad (5.45)$$

It follows from (5.33), (5.35), (5.42), and (5.44) that

$$\lim_{n\to\infty} \frac{2^n(n!)^2}{(2n)!}\left(\frac{z}{\sqrt{n+1}}\right)^n P_n\left(\frac{\sqrt{n+1}}{z}\right) = e^{-z^2/4}, \quad (5.46)$$

the convergence being uniform on every set $|z| \leqslant R$.

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